数理解析研究所講究録
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進行波に沿った線型化固有値問題への TOPOLOGICAL なアプローチ

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1. Introduction

We treat the following FitzHugh-Nagumo equations as an example.

$$\begin{cases} u_t = u_{xx} + f(u) - w \\ w_t = \varepsilon(u - \gamma w), \end{cases} \tag{1.1}$$

where $x, t \in \mathbb{R}$ and $u(x,t), w(x,t) \in \mathbb{R}$, and $1 \gg \varepsilon > 0, \gamma > 0$ are parameters. In the system, the non-linear term $f(u)$ is assumed to be a smooth cubic-like function of u satisfying the conditions below.

(1) $f(0) = f(a) = f(1) = 0$, for some constant a with $0 < a < 1$.
(2) $f'(0) < 0$ and $f'(1) < 0$.
(3) $f(u) > 0$ if $u \in (-\infty, 0) \cup (a, 1)$ and $f(u) < 0$ if $u \in (0, a) \cup (1, +\infty)$.
(4) $\int_0^1 f(u)du > 0$.

In this paper we shall restrict our attention to large $\gamma > 0$ so that the system (1.1) has three spatially homogeneous stationary solutions $(u, w) \equiv (u_1, w_1) := (0, 0)$, $(u_\dagger, w_\dagger)$ and (u_2, w_2). Here u_* and w_* ($* = 1, 2$ or $\dagger$) are constants which satisfy

$$\begin{cases} f(u_*) - w_* = 0 \\ u_* - \gamma w_* = 0, \end{cases} \quad i = 1, 2 \text{ or } \dagger$$
$$0 = u_1 < u_\dagger < u_2 < 1.$$

The system (1.1) has spatial solutions called travelling waves which are explained in the sequel.

Let $\xi = x + ct$ be a moving frame for some constant $c > 0$, then in (ξ, t) coordinate, (1.1) is expressed as

$$\begin{cases} u_t = u_{\xi\xi} - cu_\xi + f(u) - w \\ w_t = -cw_\xi + \varepsilon(u - \gamma w). \end{cases} \tag{1.2}$$

A travelling wave solution $(u(x,t), w(x,t)) = (u(\xi), w(\xi))$ of (1.1) at velocity c is a steady state solution of (1.2) *i.e.* $(u(\xi), w(\xi))$ satisfies the equations

$$\begin{cases} u_{\xi\xi} - cu_\xi + f(u) - w = 0 \\ -cw_\xi + \varepsilon(u - \gamma w) = 0. \end{cases} \tag{1.3}$$

Often (1.3) is treated in form of first order equations,

$$\begin{cases} u' = v \\ v' = cv - f(u) + w \qquad (\,' = \dfrac{d}{d\xi}) \\ w' = \dfrac{\varepsilon}{c}(u - \gamma w). \end{cases} \tag{1.4}$$

This system shall be simply written as

$$z' = X(z; \mu)$$

where $z = (u, v, w)$ and $\mu = (\gamma, c; \varepsilon)$. $a_1 := (u_1, 0, w_1) = (0, 0, 0)$ and $a_2 := (u_2, 0, w_2)$ are equilibria of (1.4).

It is well known that (1.4) has a heteroclinic solution $z_1(\xi)$ from a_1 to a_2 ($z_2(\xi)$ from a_2 to a_1) for certain parameter values. This solution corresponds to a travelling wave of (1.1) which satisfies

$$\lim_{\xi \to +\infty} z_1(\xi) = a_1, \lim_{\xi \to -\infty} z_1(\xi) = a_2$$

$$\left(\lim_{\xi \to +\infty} z_2(\xi) = a_2, \lim_{\xi \to -\infty} z_2(\xi) = a_1, \text{ respectively} \right).$$

This wave is called travelling front, or simple front in the terminology in Deng [6] (travelling back or simple back respectively). Deng [6] proved that for certain parameter value $\mu_0 = (\gamma_0(\varepsilon), c_0(\varepsilon), \varepsilon)$, the system (1.4) has heteroclinic solutions z_1 and z_2 simultaneously forming what is called a heteroclinic loop $\Gamma = (\cup \Gamma_i) \cup (\cup a_i)$, $\Gamma_i = \{z_i(\xi) | \xi \in \mathbb{R}\}$. Furthermore, there is a sequence of N-heteroclinic solutions $\{z_{(N),1}(\xi)\}_{N=1}^{\infty}$ from a_1 to a_2 ($\{z_{(N),2}(\xi)\}_{N=1}^{\infty}$ from a_2 to a_1) which correspond to travelling waves called N-fronts (N-backs respectively) bifurcating from the heteroclinic loop, together with homoclinic solutions to a_1 and a_2 which correspond to travelling pulses (simple impulses, in Deng's terminology). Here an N-heteroclinic solution from a_1 to a_2 (from a_2 to a_1) is a heteroclinic solution from a_1 to a_2 (from a_2 to a_1) which rounds N-times and a half in some tubular neighborhood of the heteroclinic loop.

We are concerned with the stability of these travelling waves. Eigenvalue problem for (1.2) along the travelling wave under study is often investigated to determine the stability of the wave, because stability for the linear problem implies the same for the full nonlinear problem. See Evans [7].

The linear stability is established as follows. Consider the linearization of (1.2) along the travelling wave $(u(\xi), v(\xi))$ which is under consideration,

$$\begin{cases} P_t = P_{\xi\xi} - cP_\xi + Df\,(u(\xi))\,P - R \\ R_t = -cR_\xi + \varepsilon(P - \gamma R). \end{cases} \tag{1.5}$$

The right hand side of (1.5) defines a densely defined closed operator

$$L \begin{pmatrix} P \\ R \end{pmatrix} := \begin{pmatrix} P_{\xi\xi} - cP_\xi + Df\,(u(\xi))\,P - R \\ -cR_\xi + \varepsilon(P - \gamma R) \end{pmatrix}$$

on the space $BU(\mathbb{R}, \mathbb{R}^2) := \{\phi\colon \mathbb{R} \to \mathbb{R}^2 |$ bounded uniformly continuous$\}$ with supremum norm. Then, following fact is well known (Evans [7], Bates and Jones [4]).

Fact.
Let $\sigma(L)$ be the spectrum of L, then the travelling wave $(u(\xi), v(\xi))$ is stable if the conditions below are satisfied.

(1) *There exist $\beta < 0$ so that $\sigma(L) \setminus \{0\} \subset \{\lambda | Re\lambda < \beta\}$.*
(2) 0 *is a simple eigenvalue.*

Remark 1.1.

(1) *L has as an eigenvalue* 0 *corresponding to spatial translation of the wave.*
(2) *Concerning a wave which connect stable steady states, there exists $\beta < 0$ so that $\sigma(L) \cap \{\lambda | Re\lambda > \beta\}$ consists of only eigenvalues with finite multiplicity. (See Jones* [10] *for FitzHugh-Nagumo equations, Henry* [9] *for general cases.)*

Thus the stability problem amount to proving nonexistence of eigenvalue of L other than zero real part of which is equal or grater than 0, and also the simplicity of the zero eigenvalue.

2. Basic Ideas

This section is devoted to a brief sketch of basic ideas to investigate the eigenvalue problem associated with N-fronts (N-back).

The eigenvalue problem

$$\begin{cases} P_{\xi\xi} - cP_\xi + Df\,(u(\xi))\,P - R = \lambda P \\ \qquad\qquad -cR_\xi + \varepsilon(P - \gamma R) = \lambda R. \end{cases} \tag{2.1}$$

can be regarded as a system of second order linear ordinary differential equations. This system shall be also treated in the following form of first order system,

$$\begin{cases} P' = Q \\ Q' = cQ - Df\,(u(\xi))\,P + \lambda P + R \qquad (\ ' = \dfrac{d}{d\xi}) \\ R' = \dfrac{\varepsilon}{c}(P - \gamma R) - \dfrac{\lambda}{c}R, \end{cases} \tag{2.2}$$

or simply

$$p' = A\,(u(\xi); \lambda)\,p,$$

where $p = (P, Q, R)$ and

$$A\,(u(\xi); \lambda) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda - Df\,(u(\xi)) & c & 1 \\ \frac{\varepsilon}{c} & 0 & -\frac{1}{c}(\varepsilon\gamma + \lambda) \end{pmatrix}.$$

For $Re\lambda > \beta$ the matrices

$$A_{\pm}(\lambda) := A(a_{i\pm}; \lambda) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda - Df(a_{i\pm}) & c & 1 \\ \frac{\varepsilon}{c} & 0 & -\frac{1}{c}(\varepsilon\gamma + \lambda) \end{pmatrix}$$

in both ends ($\xi \to \pm\infty$) of (2.2) have one unstable eigenvalue $\nu^u_{i\pm}(\lambda) > 0$ and two stable ones $-\nu^{ss}_{i\pm} < -\nu^s_{i\pm} < 0$, where we assume that $\lim_{\xi \to \pm\infty} (u(\xi), w(\xi)) = a_{i\pm}$. This means (2.2) has one solution $p^1(\xi; \lambda)$ up to multiplication of non-zero constant which is bounded as $\xi \to -\infty$ and two independent solutions $p^2(\xi; \lambda), p^3(\xi; \lambda)$ up to non-trivial linear combination of them which are bounded as $\xi \to +\infty$.

As the eigenvalue problem (2.1) is examined on the function space $BU(\mathbb{R}, \mathbb{R}^2)$, λ is an eigenvalue if and only if (2.2) has a non-trivial bounded solution. When above observation is taken into account, this is equivalent to linear dependence of $p^1(\xi : \lambda)$ and $p^2(\xi : \lambda)$ and $p^3(\xi : \lambda)$ *i.e.* the solution $p^1(\xi : \lambda)$ which converges to 0 as $\xi \to -\infty$ along the unstable eigenspace of $A_-(\lambda)$ also converges to 0 as $\xi \to +\infty$ along the stable eigenspace of $A_+(\lambda)$. Thus the problem becomes the problem of search for such solutions.

We shall deal with this eigenvalue problem as a full bifurcation problem.

Consider the coupled system of (1.4) and (2.2)

$$\begin{cases} z' = X(z; \mu) \\ p' = A(z; \lambda, \mu)p. \end{cases} \tag{2.3}$$

This system on $\mathbb{R}^3 \times \mathbb{C}^3$ induces a system on $\mathbb{R}^3 \times \mathbb{CP}^2$

$$\begin{cases} z' = X(z;\mu) \\ \hat{p}' = Y(z,\hat{p};\lambda,\mu) \end{cases} \tag{2.4}$$

as it is linear in p component.

Let $e_i^1(\lambda)$ $(i = 1,2)$ be an eigenvector associated with the unstable eigenvalue of $A(a_i;\lambda)$ and $e_i^2(\lambda)$ and $e_i^3(\lambda)$ be eigenvectors associated the stable eigenvalues. Further more, we assume that $e_i^2(\lambda)$ belongs to the eigenspace corresponding to the principal stable eigenvalue which is the stable eigenvalue with its real part larger than the other. The points in $\mathbb{CP}^2$ representing eigenspaces spanned by $e_i^j(\lambda)$ shall be denoted as $\hat{e}_i^j(\lambda)$. Then for each $i = 1,2$, $\{a_i\} \times \mathbb{CP}^2$ is an invariant set of (2.4), which consists of equilibria $\left(a_i, \hat{e}_i^j(\lambda)\right)$ $(j = 1,2,3)$ and heteroclinic orbits between them. For the parameter value μ at which (1.4) has a heteroclinic solution from a_{i_-} to a_{i_+}, the system (2.4) should have a heteroclinic solution from $\left(a_{i_-}, \hat{e}_{i-}^1(\lambda)\right)$ to $\left(a_{i_+}, \hat{e}_{i+}^j(\lambda)\right)$ for some j depending on λ. For generic λ this solution should be from $\left(a_{i_-}, \hat{e}_{i-}^1(\lambda)\right)$ to $\left(a_{i_+}, \hat{e}_{i+}^1(\lambda)\right)$, because $\left(a_{i_+}, \hat{e}_{i+}^1(\lambda)\right)$ is an attracting equilibrium in the invariant set $\{a_{i_+}\} \times \mathbb{CP}^2$ and the complementary repeller consists of $\left(a_{i_+}, \hat{e}_{i+}^2(\lambda)\right)$, $\left(a_{i_+}, \hat{e}_{i+}^3(\lambda)\right)$ and the heteroclinic orbits from $\left(a_{i_+}, \hat{e}_{i+}^3(\lambda)\right)$ to $\left(a_{i_+}, \hat{e}_{i+}^3(\lambda)\right)$. In fact, the existence of the solution from $\left(a_{i_-}, \hat{e}_{i-}^1(\lambda)\right)$ to $\left(a_{i_+}, \hat{e}_{i+}^2(\lambda)\right)$ or $\left(a_{i_+}, \hat{e}_{i+}^3(\lambda)\right)$ means that λ is an eigenvalue of L and vice versa.

Let μ_0 be a parameter value at which (1.4) has a heteroclinic loop consisting of heteroclinic solutions $z_1(\xi)$ from a_1 to a_2 and $z_2(\xi)$ from a_2 to a_1. Then, for $(\lambda,\mu) = (0,\mu_0)$, (2.4) has heteroclinic solutions from $(a_1, \hat{e}_1^1(0))$ to $(a_2, \hat{e}_2^2(0))$ and from $(a_2, \hat{e}_2^1(0))$ to $(a_1, \hat{e}_1^2(0))$ simultaneously. We interpret the eigenvalue problem associated with N-front wave which corresponds to N-heteroclinic solution from a_{i_-} to a_{i_+} as a bifurcation problem of finding N-heteroclinic solution of (2.4) from $\left(a_{i_-}, \hat{e}_{i-}^1(\lambda)\right)$ to $\left(a_{i_+}, \hat{e}_{i+}^2(\lambda)\right)$ or $\left(a_{i_+}, \hat{e}_{i+}^3(\lambda)\right)$.

We employ a topological approach to detect existence of such solutions.

Let us consider a scalar equation instead of (1.1):

$$u_t = u_{xx} + f(u) \tag{2.5}$$

and assume that this equation has a travelling wave $u(\xi)$ $(\xi = x + ct)$. Then the linearized eigenvalue problem associated with $u(\xi)$ becomes

$$\begin{cases} P' = Q \\ Q' = cQ - Df\left(u(\xi)\right)P + \lambda P \end{cases} \tag{2.6}$$

or

$$p' = A\left(u(\xi);\lambda\right)p$$

where $p = (P, Q)$ and

$$A\left(u(\xi);\lambda\right) = \begin{pmatrix} 0 & 1 \\ \lambda - Df\left(u(\xi)\right) & c \end{pmatrix}.$$

If $u(\xi)$ approaches to stable steady state solutions $u_{\pm}$ as $\xi \to \pm\infty$ *i.e.* $\lim_{\xi\to\pm\infty} u(\xi) = u_{\pm}$, then each of

$$A_{\pm}(\lambda) := A\left(u_{\pm},\lambda\right) = \begin{pmatrix} 0 & 1 \\ \lambda - Df\left(u_{\pm}\right) & c \end{pmatrix}$$

has one stable eigenvalue and one unstable one if $Re\lambda > \beta$ for some $\beta < 0$. From now on, let us restrict our attention to real eigenvalues, that is, $\beta < \lambda$ is real and (2.6) is regarded as a system on $\mathbb{R}^2$. Then (2.6) induces an equation on $\mathbb{RP}^1 \cong \mathbb{S}^1$:

$$\hat{p}' = Y\left(u(\xi),\hat{p};\lambda\right). \tag{2.7}$$

For each of $-$ or $+$, let $e^u_{\pm}$ be an unstable eigenvector associated with the unstable eigenvalue of $A_{\pm}(\lambda)$ and $e^s_{\pm}$ be a stable one. Then (2.7) has a solution $\hat{p}(\xi;\lambda)$ which satisfies $\lim_{\xi\to\pm\infty}\hat{p}(\xi;\lambda) = \hat{e}^u_{\pm}$ if λ is not an eigenvalue, where $\hat{e}^u_{\pm}$ or $\hat{e}^s_{\pm}$ are the points on $\mathbb{RP}^1$ corresponding to $e^u_{\pm}$ or $e^s_{\pm}$.

Now an index which detects real eigenvalues of the eigenvalue problem (2.6) shall be defined. (*cf.* N. [12])

Let us define a map $\mathfrak{g}\colon \partial\left([\lambda_1,\lambda_2]\times[-1,1]\right) \cong \mathbb{S}^1 \to \mathbb{RP}^1$ as

$$\mathfrak{g}(\lambda,\tau) = \begin{cases} \hat{e}^u_{\pm}(\lambda) & \lambda\in[\lambda_1,\lambda_2], \quad \tau = \pm 1 \\ \hat{p}\left(\log\left(\dfrac{1+\tau}{1-\tau}\right);\lambda_i\right) & \lambda = \lambda_i \quad (i=1,2), \quad \tau\in(-1,1) \end{cases} \tag{2.8}$$

for $\beta < \lambda_1 < \lambda_2$ which are not eigenvalues. Then $\mathfrak{g}$ is continuous and induces an homomorphism $\mathfrak{g}_*\colon H_1\left(\partial\left([\lambda_1,\lambda_2]\times[-1,1]\right)\right) \to H_1\left(\mathbb{RP}^1\right)$.

If there is no eigenvalue in the interval $[\lambda_1,\lambda_2]$, then the isomorphism $\mathfrak{g}_*$ is trivial. This is because in such case $\mathfrak{g}$ can be naturally extended to a map defined on whole $[\lambda_1,\lambda_2]\times[-1,1]$ and thus $\mathfrak{g}$ is homotopic to a map into one point. More over if $\mathfrak{g}_*(1) = n$ then there are at least n eigenvalues in the interval $[\lambda_1,\lambda_2]$. Here $H_1\left(\partial\left([\lambda_1,\lambda_2]\times[-1,1]\right)\right)$ and $H_1\left(\mathbb{RP}^1\right)$ are identified with $\mathbb{Z}$.

Unfortunately, this index is not effective for FitzHugh-Nagumo equations (1.1) as $H_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$ and thus we can only detect the parity of eigenvalues in the interval $[\lambda_1,\lambda_2]$.

In the sequel we shall construct an index for N-front solution of FitzHugh-Nagumo equations which is a modification of this index.

3. Construction of the index

In this section we construct an index for small $|\lambda|$ for the N-front wave bifurcating from coexisting simple front and simple back which is analogous to the one explained in the previous section. The strategy is to construct a subset $\Omega \subset \mathbb{RP}^2$ with $H_1(\Omega) = \mathbb{Z}$ on which the eigenvalue problem (2.4) can be restricted.

Let B_i be a small neighborhood of the equilibrium a_i $(i = 1, 2)$ in which the system (1.4) is linear and N_i be a neighborhood of the orbit $\Gamma_i = \{z_i(\xi) | \xi \in \mathbb{R}\}$ for $\mu = \mu_0$. Then $\mathcal{N} := (\cup B_i) \cup (\cup N_i)$ is a tubular neighborhood of the heteroclinic loop Γ consists of Γ_i and a_i $(i = 1, 2)$.

From now on we construct a suitable coordinate for the vector bundle $\mathcal{N} \times \mathbb{R}^3 \subset \mathbb{R}^3 \times \mathbb{R}^3$.

Let us define the directions of $e_i^k(0)$ $(i = 1, 2$ and $k = 1, 2, 3)$ so that

$$\lim_{\xi \to -\infty} \frac{z_i(\xi) - a_i}{|z_i(\xi) - a_i|} = e_i^1(0) \tag{3.1}$$

$$\lim_{\xi \to +\infty} \frac{z_j(\xi) - a_i}{|z_j(\xi) - a_i|} = -e_i^2(0) \qquad (i \neq j) \tag{3.2}$$

and the triple $(e_i^1(0), e_i^2(0), e_i^3(0))$ forms a right-handed system. This choice can be made because $A(a_i; \lambda)$ has only one unstable eigenvalue and the heteroclinic loop Γ is non-degenerate. (See Deng [6].) Then the following holds.

Lemma 3.1.
There exist solutions $f^1(\xi)$, $f^2(\xi)$ and $f^3(\xi)$ of the system $p' = A(z_1(\xi); 0, \mu_0)$ which satisfy

$$\lim_{\xi \to -\infty} f^1(\xi) e^{-\nu_1^u(0)\xi} = e_1^1(0) \qquad \lim_{\xi \to +\infty} f^1(\xi) e^{\nu_2^s(0)\xi} = -\frac{1}{\phi_0^1} e_2^2(0) \tag{3.3}$$

$$\lim_{\xi \to -\infty} f^2(\xi) e^{\nu_1^s(0)\xi} = e_1^2(0) \qquad \lim_{\xi \to +\infty} f^2(\xi) e^{-\nu_2^u(0)\xi} = -\frac{1}{\phi_0^2} e_2^1(0) \tag{3.4}$$

$$\lim_{\xi \to -\infty} f^3(\xi) e^{\nu_1^{ss}(0)\xi} = e_1^3(0) \qquad \lim_{\xi \to +\infty} f^3(\xi) e^{\nu_2^{ss}(0)\xi} = -\frac{1}{\phi_0^3} e_2^3(0) \tag{3.5}$$

for some positive constants ϕ_0^1 ϕ_0^2 and ϕ_0^3 and there exist solutions $g^1(\xi)$, $g^2(\xi)$ and

$g^3(\xi)$ of the system $p' = A\left(z_2(\xi); 0, \mu_0\right)$ which satisfy

$$\lim_{\xi\to-\infty} g^1(\xi)e^{-\nu_2^u(0)\xi} = e_2^1(0) \qquad \lim_{\xi\to+\infty} g^1(\xi)e^{\nu_1^s(0)\xi} = -\frac{1}{\psi_0^1}e_1^2(0) \tag{3.6}$$

$$\lim_{\xi\to-\infty} g^2(\xi)e^{\nu_2^s(0)\xi} = e_2^2(0) \qquad \lim_{\xi\to+\infty} g^2(\xi)e^{-\nu_1^u(0)\xi} = -\frac{1}{\psi_0^2}e_1^1(0) \tag{3.7}$$

$$\lim_{\xi\to-\infty} g^3(\xi)e^{\nu_2^{ss}(0)\xi} = e_2^3(0) \qquad \lim_{\xi\to+\infty} g^3(\xi)e^{\nu_1^{ss}(0)\xi} = -\frac{1}{\psi_0^3}e_1^3(0). \tag{3.8}$$

for some positive constants ψ_0^1, ψ_0^2 and ψ_0^3.

Let $\phi^i(\xi)$ and $\psi^i(\xi)$ $(i = 1, 2, 3)$ be smooth positive functions satisfying the following condition.

$$\phi^1(\xi) = \begin{cases} e^{-\nu_1^u(0)\xi} & (\xi \le -\xi_1) \\ \phi_0^1 e^{\nu_2^s(0)\xi} & (\xi \ge \xi_1) \end{cases} \qquad \psi^1(\xi) = \begin{cases} e^{-\nu_2^u(0)\xi} & (\xi \le -\xi_2) \\ \psi_0^1 e^{\nu_1^s(0)\xi} & (\xi \ge \xi_2) \end{cases} \tag{3.9}$$

$$\phi^2(\xi) = \begin{cases} e^{\nu_1^s(0)\xi} & (\xi \le -\xi_1) \\ \phi_0^2 e^{-\nu_2^u(0)\xi} & (\xi \ge \xi_1) \end{cases} \qquad \psi^2(\xi) = \begin{cases} e^{\nu_2^s(0)\xi} & (\xi \le -\xi_2) \\ \psi_0^2 e^{-\nu_1^u(0)\xi} & (\xi \ge \xi_2) \end{cases} \tag{3.10}$$

$$\phi^3(\xi) = \begin{cases} e^{\nu_1^{ss}(0)\xi} & (\xi \le -\xi_1) \\ \phi_0^3 e^{\nu_2^{ss}(0)\xi} & (\xi \ge \xi_1) \end{cases} \qquad \psi^3(\xi) = \begin{cases} e^{\nu_2^{ss}(0)\xi} & (\xi \le -\xi_2) \\ \psi_0^3 e^{\nu_1^{ss}(0)\xi} & (\xi \ge \xi_2), \end{cases} \tag{3.11}$$

for some constants $\xi_1, \xi_2 > 0$. And put

$$F^1(\xi) = \phi^1(\xi)f^1(\xi) \qquad G^1(\xi) = \psi^1(\xi)g^1(\xi) \tag{3.12}$$

$$F^2(\xi) = \phi^2(\xi)f^2(\xi) \qquad G^2(\xi) = \psi^2(\xi)g^2(\xi) \tag{3.13}$$

$$F^3(\xi) = \phi^3(\xi)f^3(\xi) \qquad G^3(\xi) = \psi^3(\xi)g^3(\xi) \tag{3.14}$$

then

$$\lim_{\xi\to+\infty} F^1(\xi) = -\lim_{\xi\to-\infty} G^2(\xi) \tag{3.15}$$

$$\lim_{\xi\to+\infty} F^2(\xi) = -\lim_{\xi\to-\infty} G^1(\xi) \tag{3.16}$$

$$\lim_{\xi\to+\infty} F^3(\xi) = -\lim_{\xi\to-\infty} G^3(\xi) \tag{3.17}$$

and

$$\lim_{\xi\to+\infty} G^1(\xi) = -\lim_{\xi\to-\infty} F^2(\xi) \tag{3.18}$$

$$\lim_{\xi\to+\infty} G^2(\xi) = -\lim_{\xi\to-\infty} F^1(\xi) \tag{3.19}$$

$$\lim_{\xi\to+\infty} G^3(\xi) = -\lim_{\xi\to-\infty} F^3(\xi). \tag{3.20}$$

By choosing small B_1, we can assume that $F^2(\xi)$ is arbitrary near $e_1^2(0)$ and $F^3(\xi)$ is near $e_1^3(0)$ when $z_1(\xi)$ is in B_1 and $G^1(\xi)$ and $G^2(\xi)$ are near $-e_1^2(0)$ and $-e_1^1(0)$ when $z_2(\xi)$ is in B_1. Moreover, as the system (1.4) is linear in B_1, $F^1(\xi) = e_1^1(0)$ for $z_1(\xi) \in B_1$ and $G^3(\xi) = -e_1^3(0)$ for $z_2(\xi) \in B_1$. Then, we modify $F^k(\xi)$ and $G^k(\xi)$ so that $F^k(\xi) = e_1^k(0)$ if $z_1(\xi) \in B_1$ and $G^k((\xi) = -e_1^k(0)$ if $z_2(\xi) \in B_1$ ($k = 1, 2, 3$) and are still smooth. Same goes for B_2.

With these $F^k(\xi)$ and $G^k(\xi)$ we define a trivialization of the vector bundle $\Phi\colon T\mathbb{R}^3|_\Gamma \to \Gamma \times \mathbb{R}^3$ by

$$\Phi(z, v) := (z, v^1, v^2, v^3) \tag{3.21}$$

for

$$v = \begin{cases} v^1 e_1^1 + v^2 e_1^2 + v^3 e_1^3 & \text{if } z = a_1 \\ -v^1 e_2^1 - v^2 e_2^2 - v^3 e_2^3 & \text{if } z = a_2 \\ v^1 F^1(\xi) + v^2 F^2(\xi) + v^3 F^3(\xi) & \text{if } z = z_1(\xi) \\ -v^1 G^1(\xi) - v^2 G^2(\xi) - v^3 G^3(\xi) & \text{if } z = z_2(\xi). \end{cases} \tag{3.22}$$

The modification of $F^k(\xi)$ and $G^k(\xi)$ above makes it possible to extend Φ to a trivialization over $\Gamma \cup B_1 \cup B_2$: $\Phi\colon\ T\mathbb{R}^3|_{\Gamma\cup B_1\cup B_2} \to \{\Gamma \cup B_1 \cup B_2\} \times \mathbb{R}^3$ by putting

$$\Phi(z, v) := (z, v^1, v^2, v^3) \tag{3.23}$$

for

$$v = \begin{cases} v^1 e_1^1 + v^2 e_1^2 + v^3 e_1^3 & \text{if } z \in B_1 \\ -v^1 e_2^1 - v^2 e_2^2 - v^3 e_2^3 & \text{if } z \in B_2 \end{cases} \tag{3.24}$$

After that we extend Φ arbitrarily to a smooth trivialization $\Phi\colon\ T\mathbb{R}^3|_{\mathcal{N}} \to \mathcal{N} \times \mathbb{R}^3$.

Then, though $D\Phi\colon T\,(T\mathbb{R}^3|_{\mathcal{N}}) \to T\,(\mathcal{N} \times \mathbb{R}^3)$, (2.3) is transformed into a system on $\mathcal{N} \times \mathbb{R}^3$:

$$\begin{cases} z' &= X(z;\mu) \\ q' &= B(z;\lambda,\mu)q \end{cases} \qquad \text{on} \quad \mathcal{N} \times \mathbb{R}^3. \tag{3.25}$$

Again this system is projectivized:

$$\begin{cases} z' &= X(z;\mu) \\ \hat{q}' &= Z(z,\hat{q};\lambda,\mu) \end{cases} \qquad \text{on} \quad \mathcal{N} \times \mathbb{RP}^2. \tag{3.26}$$

Consider a subspace $\mathcal{N} \times \Omega \subset \mathcal{N} \times \mathbb{RP}^2$ where Ω is a tubular neighborhood of $\mathbb{RP}^1 = \{[v^1 : v^2 : 0]\} \subset \mathbb{RP}^2$ with $H_1(\Omega) = \mathbb{Z}$.

If B_1 and B_2 are small, the modification of $F^k(\xi)$ and $G^k(\xi)$ is small and thus $\hat{\Phi}\,(z_i(\xi), z_{i\xi}(\xi))$ is in $\mathcal{N} \times \Omega$ ($i = 1, 2$). Here, $\hat{\Phi}\colon \bigcup_{z\in\mathcal{N}} \mathbb{P}\,(T_z\mathbb{R}^3) \to \mathcal{N} \times \mathbb{RP}^2$ is a map induced by $D\Phi$. Moreover, $\{a_i\} \times \mathbb{RP}^1$ is an attracting invariant set in an invariant subspace $\{a_i\} \times \mathbb{RP}^2$ for $\mu = \mu_0$ and $\lambda = 0$. Thus $B_i \times \Omega$ is forward invariant in $B_i \times \mathbb{RP}^2$ for μ near μ_0 and λ near 0.

Let $z_{(N),1}(\xi;\mu)$ be the N-heteroclinic solution of (1.4) from a_1 to a_2 for $\mu \approx \mu_0$. Then in the limit of $\mu \to \mu_0$ the orbit $\Gamma_{(N)} := \left\{ z_{(N),1}(\xi;\mu) | \xi \in \mathbb{R} \right\}$ converges to Γ and $z_{(N),1}(\xi;\mu)$ stays in B_i long time whereas the length of time when $z_{(N),1}(\xi;\mu)$ stays out of B_i is bounded. This means the solution $\left(z_{(N),1}(\xi;\mu), z_{(N),1\xi}(\xi;\mu) \right)$ of (3.26) for $\lambda = 0$ stays in $\mathcal{N} \times \Omega$ if μ is near μ_0. Moreover we the following holds.

Lemma 3.2.
A solution $\left(z_{(N),1}(\xi;\mu), \hat{q}(\xi;\lambda,\mu) \right)$ of (3.26) which satisfies $\lim_{\xi \to -\infty} (z(\xi;\mu), \hat{q}(\xi;\lambda,\mu)) = \hat{\Phi}\left(a_1, \hat{e}_1^1(\lambda)\right)$ stays in $\mathcal{N} \times \Omega$ for $\mu \approx \mu_0$ and $\lambda \approx 0$.

Then we define a map

$$\mathfrak{g} \colon \partial\left([\lambda_1, \lambda_2] \times [-1,1]\right) \to \Omega \tag{3.27}$$

analogous to the map in the previous section by

$$\mathfrak{g}(\lambda,\tau) = \begin{cases} \hat{\Phi}_q\left(\hat{e}_1^1(\lambda)\right) & \lambda \in [\lambda_1,\lambda_2], \quad \tau = -1 \\ \hat{\Phi}_q\left(\hat{e}_2^1(\lambda)\right) & \lambda \in [\lambda_1,\lambda_2], \quad \tau = +1 \\ \hat{q}\left(\log\left(\frac{1+\tau}{1-\tau}\right); \lambda_i\right) & \lambda = \lambda_i \quad (i = 1,2), \quad \tau \in (-1,1) \end{cases} \tag{3.28}$$

when $\lambda_1 < \lambda_2$ are not eigenvalues with $|\lambda_1|$, $|\lambda_2|$ small and $\mu \approx \mu_0$ where $\hat{\Phi} = \left(\hat{\Phi}_z, \hat{\Phi}_q\right)$. This map induces a homomorphism $\mathfrak{g}_* \colon H_1\left(\partial\left([\lambda_1,\lambda_2] \times [-1,1]\right)\right) \to H_1(\Omega) \cong \mathbb{Z}$. Then $\mathfrak{g}_*(1)$ counts the number of eigenvalues in $[\lambda_1, \lambda_2]$.

4. The result

We can prove the following based on the strategy explained above.

Theorem (N. [14]).
Assume that the system (1.4) is linear in some small neighborhoods of equilibria a_i $(i = 1,2)$, then the N-front (N-back) bifurcating from the heteroclinic loop at $\mu = \mu_0\left(\gamma_0(\varepsilon), c_0(\varepsilon), \varepsilon\right)$ is stable for $\mu \approx \mu_0$.

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